Problem 5)

a) $\Gamma(1) = \int_0^\infty e^{-y} dy = -e^{-y}|_0^\infty = 1.$

b) $\Gamma(2) = \int_0^\infty y e^{-y} dy = -y e^{-y}|_0^\infty + \int_0^\infty e^{-y} dy = 0 - 0 + 1 = 1.$ ← Integration by parts

c) $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy = -y^{x-1} e^{-y}|_0^\infty + \int_0^\infty (x-1) y^{x-2} e^{-y} dy$ ← Integration by parts

$$= 0 - 0 + (x-1) \int_0^\infty y^{x-2} e^{-y} dy = (x-1)\Gamma(x-1).$$

d) For an integer value of x greater than or equal to 2, namely, $x = n \geq 2$, we will have

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots$$
$$= (n-1)(n-2)(n-3)\cdots 3 \times 2 \times 1 \times \Gamma(1) = (n-1)!.$$

e) $\Gamma(½) = \int_0^\infty y^{-½} e^{-y} dy \underset{\uparrow}{=} 2\int_0^\infty e^{-x^2} dx \underset{\uparrow}{=} \sqrt{\pi}.$

Change of variable: $x = y^{½} \rightarrow dx = ½ y^{-½} dy$

HW Problem 4

f) Using the results of parts (c) and (e) above, we may write for $x = n + ½$ where $n \geq 1$,

$$\Gamma(n + ½) = \left(n - \tfrac{1}{2}\right)\Gamma\left(n - \tfrac{1}{2}\right) = \left(n - \tfrac{1}{2}\right)\left(n - \tfrac{3}{2}\right)\Gamma\left(n - \tfrac{3}{2}\right) = \cdots$$
$$= \left(n - \tfrac{1}{2}\right)\left(n - \tfrac{3}{2}\right)\cdots \tfrac{3}{2} \times \tfrac{1}{2} \times \Gamma\left(\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$
